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ON THE EXACT AND APPROXIMATE LINEAR THEORY OF VERTICALLY PROPAGATING PLANETARY ROSSBY WAVES FORCED AT A SPHERICAL LOWER BOUNDARY

ROBERT E. DICKINSON

Massachusetts Institute of Technology, Cambridge, Mass.

ABSTRACT

An approximate linearized model for the analysis of low frequency transient and stationary planetary scale atmospheric waves is derived. The problem of stationary waves forced at the lower boundary is solved exactly for an atmosphere in constant rotation by making use of the recent tidal theory analysis of Longuet-Higgins, and this solution is then compared with the solution of the approximate model equation. The approximate model is found to describe the Rossby wave modes with little error away from the Tropics. The eigenvalues of the approximate model are in good agreement with the eigenvalues of the exact model with the exception of eigenvalues of the lowest latitudinal modes. It is concluded that the model will be suitable for the purpose of linear theoretical analysis of the vertical propagation of planetary Rossby waves in the presence of zonal wind shears. Assuming an atmosphere in constant angular rotation and assuming westerly zonal wind velocities of the magnitude of the maximum winds observed in the midwinter stratospheric jet, there will always be two or more planetary wave modes that can propagate vertically. The constant angular wind velocity model is used together with the amplitude of observed stationary planetary waves in the winter lower stratosphere to predict the magnitude of planetary waves at the meteor wind level. Because the amplitude of the eddy winds so predicted exceeds observed values by at least an order of magnitude, we infer that horizontal wind shears and possibly also diabatic damping need be considered for the description of the propagation of planetary waves from the troposphere to the lower thermosphere.

1. INTRODUCTION

Stationary and low frequency-transient eddy winds of planetary scale dimensions are excited in the earth's atmosphere as a consequence of vertical motions forced by continent-ocean thermal contrast or by flow of zonal winds over the planetary scale components of the earth's topography (cf. Saltzman [17], Sankar-Rao [20], and earlier references given in these papers) and are modified by interaction with cyclone scale motions (cf. Saltzman and Fleischer [18]). As a result of the earth's variable vertical component of vorticity these winds can propagate as waves, and hence a source confined to one latitude at the ground may excite motions at great heights and in both hemispheres. The waves associated with such motions of time scales of several days or longer are strongly refracted by variable zonal winds. There are turning points of the nondissipative wave equation along lines of zero zonal wind and also along lines defined by some strength of a westerly zonal wind, so that there is wave propagation only in weak westerly zonal winds (Charney and Drazin [3]). Hence near the solstices vertical propagation in the stratosphere and mesosphere will be primarily confined to the polar and equatorial wave guides in regions where such weak westerlies do occur (Dickinson [4]). Present observational evidence demonstrates the realization of such waves in the polar wave guide to the present upper limit of synoptic meteorological data at the stratopause (Finger, Woolf, and Anderson [6]), while rocket grenade data (Nordberg et al. [14]) and meteor wind data (Newell and Dickinson [13]) indicate the presence of planetary scale eddy motions at least up to the mesopause.

The detailed theoretical explanation of the observed motions will depend on satisfactory resolution of two questions: 1) what are the energy sources for the observed eddy motions? 2) what are the transmission characteristics of the atmosphere for arbitrary sources? Because observed motion phenomena may be rather far removed in space and time from the actual energy sources, it would appear that the latter question need be answered first. For this purpose, it is necessary to select a suitable mathematical model for such atmospheric motions. Quantitatively accurate analysis will undoubtedly require numerical models, but it seems first desirable to study analytic models to understand better the relevant dynamical processes and to obtain preliminary rough estimates of the magnitudes of various effects. Because of the wide latitudinal extent of the motions of interest, it appears that a highly desirable property of such models would be that they be valid on a sphere. On the other hand, because of the apparent great importance of horizontal and vertical shears, it is also desirable that the models not be completely intractable when shears are included.

One purpose of the present paper then is to formulate an approximate theoretical model which appears to satisfy the above criteria. In order to obtain the equations for this model, we follow the standard procedure of deriving a vorticity equation and a divergence equation, then using the divergence equation to express geopotential in terms of a stream function. The approximate spherical earth model obtained here is similar to that discussed by Kuo [8], except for the linearization of the present paper. There is no indication in earlier works of the errors that might be expected to result from using the approximate model to study planetary waves.

The equations defining the model are derived by means of a scale analysis valid away from the Equator and for assumed scales appropriate to disturbances in the winter upper stratosphere. In order to verify the validity of the approximations made in deriving the proposed model, we investigate the solution to the problem of the vertical propagation of stationary waves forced at a spherical bottom boundary for an atmosphere in constant angular rotation. It is now possible, following the definitive analytic and numerical tidal theory study by Longuet-Higgins [10], hereafter referred to as LH, to solve *exactly* for this special case the linearized vertical propagation problem. Comparison is then made between the *exact solution* for an atmosphere in constant angular rotation and the solution given by the approximate model. By this means we establish for middle and high latitudes that the terms neglected in order to derive the approximate model can be omitted with errors negligible compared to other uncertainties that will arise in any realistic application of the theory to the atmosphere.

Our discussion here of the problem of vertical propagation for an atmosphere in constant rotation is also intended to serve as the introduction to a general linear theory of vertical propagation in an atmosphere with arbitrary horizontal and vertical shears. The present paper, Dickinson [4], and further studies in progress extend in various ways to more realistic models, the analysis of Charney and Drazin [3]. We seek to gain further insight into the basic dynamics of planetary wave motions and to provide a conceptual framework on the one hand for related observational studies and on the other hand for the formulation and solution of numerical models for such motions.

Because of the current dearth of hemispheric data on winds and temperatures above 10 mb., the construction of numerical models is now essential to the computation of the vertical motions associated with planetary scale geostrophic eddy motions in the mesosphere and above. These vertical motions are presently required for the interpretation of the distribution of various atmospheric trace substances at these levels.

2. FORMULATION OF THE MODEL

The following basic notation is used: $z=\log\,(p_0/p)$ where p=presssure, $p_0=10^3$ mb., u=eastward velocity, v=northward velocity, $w=dz/dt$, h=geopotential height, T=temperature, t=time, λ=longitude, φ=latitude, Ω=frequency of earth's rotation, a=radius of the earth. Let

$$Z=2\Omega\sin\varphi+(a\cos\varphi)^{-1}(\partial v/\partial\lambda-\partial/\partial\varphi(u\cos\varphi))$$

be the vertical component of absolute vorticity. Then the primitive equations on a spherical earth may be written

$$\frac{\partial u}{\partial t}-Zv+w\frac{\partial u}{\partial z}+\frac{g}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(h+\frac{u^2+v^2}{2g}\right)=-F^{(x)}\qquad(1)$$

$$\frac{\partial v}{\partial t}+Zu+w\frac{\partial v}{\partial z}+\frac{g}{a}\frac{\partial}{\partial\varphi}\left(h+\frac{u^2+v^2}{2g}\right)=-F^{(y)}\qquad(2)$$

$$\frac{\partial T}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial T}{\partial\lambda}+\frac{v}{a}\frac{\partial T}{\partial\varphi}+\left(\frac{\partial T}{\partial z}+\kappa T\right)w=Q/c_p\qquad(3)$$

$$\frac{\partial u}{a\cos\varphi\,\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(v\cos\varphi)+\frac{\partial w}{\partial z}-w=0\qquad(4)$$

$$g\frac{\partial h}{\partial z}=RT\qquad(5)$$

which are the equations of motion, thermodynamics, continuity, and hydrostatic balance, respectively. We have used $F^{(x)}$, $F^{(y)}$ to denote the forces in the λ and φ directions, and Q to denote the rate of heating per unit mass. Also R=gas constant, c_p=specific heat at constant p, $\kappa=R/c_p$. The assumption of hydrostatic balance has been made to derive (1)–(5). Our development is similar to the usual formulation of dynamic meteorology (cf. Phillips [15]).

Averages of a dependent variable over longitude and latitude and deviations from these averages are defined by

$$\overline{(\ )}=\frac{1}{2\pi}\int_0^{2\pi}(\ )d\lambda;$$

$$\overline{\overline{(\ )}}=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\overline{(\ )}\cos\varphi\,d\varphi;$$

$$(\ )'=(\ )-\overline{(\ )};$$

$$\overline{(\ )}''=\overline{(\ )}-\overline{\overline{(\ )}}$$

so we have the decomposition

$$(\ )=\overline{\overline{(\ )}}+\overline{(\ )}''+(\ )'.$$

In order to determine which terms of the *exact* nonlinear equations may be neglected, we shall make the scaling assumptions

$$T'\sim\delta\overline{T} \qquad u',\, v',\, \overline{v}\sim\delta\overline{u}$$

where δ is a small parameter.

Now for the purpose of simplification of the analysis, we shall introduce nondimensional variables by setting

$$\left.\begin{array}{lll} t=\tilde{t}/2\Omega & u'=2\Omega a\tilde{u} & \overline{u}=2\Omega aU \\ v'=2\Omega a\tilde{v} & w'=2\Omega\tilde{w} & gh'=(2\Omega a)^2\tilde{h} \\ RT'=(2\Omega a)^2\tilde{T} & & \kappa Q'=(2\Omega)^3a^2\tilde{Q} \\ F^{(x)}{}'=(2\Omega)^2a\tilde{F}^{(x)} & & F^{(y)}{}'=(2\Omega)^2a\tilde{F}^{(y)}. \end{array}\right\} \quad (6)$$

We define a nondimensional static stability $\tilde{S}$ by

$$\tilde{S}=(2\Omega a)^{-2}R(\kappa\overline{T}+\overline{T}_z). \quad (7)$$

Note the thermal wind equation

$$-R\overline{T}''_\varphi=(2\Omega a)^2\sin\varphi\, U_z. \quad (8)$$

We shall use further notation $\sigma=\cos\varphi$, $\mu=\sin\varphi$, and take $\tilde{Z}=[\mu-\sigma^{-1}(\sigma U)_\varphi]$ to be the zonally averaged vorticity. We use (7)–(8) together with (5) and (6) and our scaling assumptions to write (1)–(4) in nondimensional, linearized form as

$$\left(\frac{\partial}{\partial\tilde{t}}+\frac{U}{\sigma}\frac{\partial}{\partial\lambda}\right)\tilde{u}-\tilde{Z}\tilde{v}+U_z\tilde{w}+\frac{\partial\tilde{h}}{\sigma\partial\lambda}=-\tilde{F}^{(x)} \quad (9)$$

$$\left(\frac{\partial}{\partial\tilde{t}}+\frac{U}{\sigma}\frac{\partial}{\partial\lambda}\right)\tilde{v}+(\tilde{Z}+\sigma(U/\sigma)_\varphi)\tilde{u}+\frac{\partial\tilde{h}}{\partial\varphi}=-\tilde{F}^{(y)} \quad (10)$$

$$\left(\frac{\partial}{\partial\tilde{t}}+\frac{U}{\sigma}\frac{\partial}{\partial\lambda}\right)\frac{\partial\tilde{h}}{\partial z}-\mu U_z v+\tilde{S}\tilde{w}=\tilde{Q} \quad (11)$$

$$\frac{\partial\tilde{u}}{\sigma\partial\lambda}+\frac{\partial(\sigma\tilde{v})}{\sigma\partial\varphi}+\frac{\partial\tilde{w}}{\partial z}-\tilde{w}=0. \quad (12)$$

Subscripts are used when convenient to indicate differentiation with respect to the subscript. The system (9)–(12) is general enough to describe any hydrostatic perturbation motion occurring on a spherical earth in the presence of geostrophically balanced mean wind.

We can follow Saltzman [16] in assuming that, if necessary, one may obtain observationally the nonlinear terms omitted in (9)–(11) and include these as *empirical forcing functions*. It has been found that the tropospheric, middle latitude, mean asymmetric motions are forced primarily by external heating and boundary effects and that the observed empirical forcing functions due to nonlinear transient terms are largest in high latitudes, ranging in magnitude from an order of magnitude smaller to as large as the external forcing functions (Saltzman and Sankar-Rao [19]). Taking the viewpoint of Saltzman, we assume that $\tilde{F}^{(x)}$, $\tilde{F}^{(y)}$, and $\tilde{Q}$ may be modified to include the nonlinear terms omitted from the left hand side.

For further discussion it is convenient to write (9) and (10) alternatively as a vorticity and divergence equation. Let $\Delta=\partial^2/\partial\varphi^2-\mu/\sigma\,\partial/\partial\varphi+\sigma^{-2}\partial^2/\partial\lambda^2$ be a Laplacian operator, $\nabla\cdot$, ∇ be divergence and gradient operators, respectively, on a unit sphere. Following Love [12] and many later authors, we introduce a stream function $\tilde{\psi}$ and velocity potential $\tilde{\phi}$ into the above system by substituting

$$\tilde{u}=\sigma^{-1}\tilde{\phi}_\lambda-\tilde{\psi}_\varphi \quad (13)$$

$$\tilde{v}=\tilde{\phi}_\varphi+\sigma^{-1}\tilde{\psi}_\lambda. \quad (14)$$

After taking curl and divergence of (9)–(10) and omitting the external forcing, we may write the result as

$$\mathcal{D}\tilde{\psi}+\mathcal{L}\tilde{\phi}-\sigma^{-1}(\sigma U_z\tilde{w})_\varphi=0 \quad (15)$$

$$\mathcal{D}\tilde{\phi}-\mathcal{L}\tilde{\psi}+\Delta\tilde{h}+(U/\sigma)_\varphi((\sigma\tilde{u})_\varphi+\tilde{v}_\lambda)+\tilde{u}(\sigma(U/\sigma)_\varphi)_\varphi +U_z\sigma^{-1}\tilde{w}_\lambda=0 \quad (16)$$

where we define the linear operators

$$\mathcal{D}=\left(\frac{\partial}{\partial\tilde{t}}+\frac{U}{\sigma}\frac{\partial}{\partial\lambda}\right)\Delta+\tilde{Z}_\varphi\frac{\partial}{\sigma\partial\lambda}$$

$$\mathcal{L}=\nabla\cdot\tilde{Z}\nabla.$$

The discussion is now restricted to planetary scale, quasi-geostrophic motions. The purpose of the remainder of this section will be to derive from the *exact* perturbation model obtained above an approximate model which is suitable for the analysis of planetary scale Rossby wave motions away from the Equator. We can gain little physical insight into the nature of Rossby waves propagating through actual zonal winds from the exact model. Furthermore, the numerical solution of the exact model appears to be a formidable task.

First let us introduce scaling assumptions essentially equivalent to those of Burger [2] and then see what modifications are necessary in order to obtain scaling appropriate to observed planetary waves in the stratosphere. The stability S is assumed to be proportional to a small parameter δ. (In the upper stratosphere S is typically 0.02 to 0.03) We shall also assume that the mean wind is smaller than $2\Omega a$ by an $O(\delta)$ factor, so that U is proportional to δ.

Let us define a stretched time variable $\hat{t}$ by taking $\hat{t}=\delta\tilde{t}$ and assume that motions do not change significantly for changes of $\hat{t}$ less than $O(1)$. That is, we are considering quasi-geostrophic motions with frequencies typically δ times the earth's rotational frequency or less. All dependent variables are taken to be of the same magnitude. Equations (15) and (16) can then be written as

$$\left.\begin{aligned}\tilde{\psi}_\lambda+\nabla\cdot\mu\nabla\tilde{\phi}+\delta A=0\\ \tilde{\phi}_\lambda-\nabla\cdot\mu\nabla\tilde{\psi}+\delta B=0\end{aligned}\right\}\tag{17}$$

where A and B are terms which for Burger scaling are $O(1)$. If we define the linear operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{D}}$ by

$$\hat{\mathcal{L}}=\left(\frac{\partial}{\partial\tilde{t}}+\frac{U}{\sigma}\frac{\partial}{\partial\lambda}\right)\Delta-(\sigma^{-1}(\sigma U)_\varphi)_\varphi\frac{\partial}{\sigma\partial\lambda}$$

$$\hat{\mathcal{D}}=-\nabla\cdot(\sigma^{-1}(\sigma U)_\varphi\nabla,$$

then we may write A and B as

$$A=\delta^{-1}\left[\hat{\mathcal{D}}\,\tilde{\psi}+\hat{\mathcal{L}}\,\tilde{\phi}-\sigma^{-1}\left(\sigma U_z\tilde{w}\right)_\varphi\right]$$

$$B=\delta^{-1}\left[\hat{\mathcal{D}}\,\tilde{\phi}-\hat{\mathcal{L}}\,\tilde{\psi}+(U/\sigma)_\varphi((\sigma\tilde{u})_\varphi+\tilde{V}_\lambda)+\tilde{u}(\sigma(U/\sigma)_\varphi)_\varphi+U_z\sigma^{-1}\tilde{w}_\lambda\right].$$

Modifications of the above Burger scaling to be used for planetary waves in the winter stratosphere are determined by the facts that: a) zonal wind systems and planetary waves are confined in latitude to less than one hemisphere rather than extending pole to pole; b) the zonal velocities are somewhat stronger than meridional velocities. A scaling consistent with these facts is obtained by taking

$$\mu=O(1),\ \partial\varphi\sim\delta^{1/2},\ \tilde{v}\sim\delta^{1/2}\tilde{u},\ \tilde{w}\sim\delta^{1/2}\tilde{u}\sim\tilde{\psi}$$

where $\partial\varphi$ formally denotes the latitudinal disturbance scale. Writing (12) as

$$\Delta\tilde{\phi}+\frac{\partial\tilde{w}}{\partial z}-\tilde{w}=0$$

we see that $\tilde{\phi}\sim\delta\psi$ so the motion is nondivergent with $O(\delta)$ error. The scaling is not valid for latitudes $\delta^{1/2}$ or less from the Equator. A more detailed β-plane analysis of the above scaling is given in Dickinson [5].

Let α denote twice the atmosphere's angular rotation

$$\alpha=2U/\cos\varphi\tag{18}$$

so that the total vertical component of vorticity due to the angular rotation of earth plus atmosphere is $(2\Omega+\alpha)\sin\varphi$. We wish to derive a system of equations that are valid for large angular rotation of the atmosphere, that is $\alpha=O(1)$, assuming, however, that the horizontal and vertical shears, $(U/\sigma)_\varphi$ and U_z, respectively, are $O(\delta)$.

The assumption that latitudinal scales are $O(\delta^{1/2})$ does not here justify a β-plane model. Also note that terms with $\cos\varphi=\sigma$ in the denominator will not necessarily be negligible when shown small by scaling since they may become large near the poles. Keeping the above criteria in mind and using the modified planetary wave scaling defined in the previous paragraph, we approximate (15) and (16) by

$$\left.\begin{aligned}\left(\frac{\partial}{\partial\tilde{t}}+\frac{U}{\sigma}\frac{\partial}{\partial\lambda}\right)\Delta\tilde{\psi}+\{\sigma-(\sigma^{-1}(\sigma U)_\varphi)_\varphi\}\frac{\partial\tilde{\psi}}{\sigma\partial\lambda}+\nabla\cdot(1+\alpha)\mu\nabla\tilde{\phi}=0\\ \Delta\tilde{h}-\nabla\cdot(1+\alpha)\mu\nabla\tilde{\psi}=0\end{aligned}\right\}\tag{19a}$$

with errors of $O(\delta^{1/2})$. To arrive at these equations, we have approximated the operator $\mathcal{L}$ by $\nabla\cdot(1+\alpha)\ \nabla$ and have omitted all terms involving velocity potential or derivatives of α in (16).

One further simplification that is permissible for $\mu=O(1)$ is to take the factor $(1+\alpha)\mu$ outside the divergence operator in the last term of the first equation and the same factor inside the gradient operator in the last term of the second equation. These approximations are made jointly so as to retain equations that are energetically consistent (Lorenz [11]) and can be justified by noting

$$\nabla\mu\cdot\nabla=\left[\frac{\mu\Delta}{\Delta\mu}1+O(\delta^{1/2}/\mu)\right]\Big\}.$$

We then have

$$\left.\begin{aligned}\left(\frac{\partial}{\partial\tilde{t}}+\frac{U}{\sigma}\frac{\partial}{\partial\lambda}\right)\Delta\tilde{\psi}+\{\sigma-(\sigma^{-1}(\sigma U)_\varphi)_\varphi\}\frac{\partial\tilde{\psi}}{\sigma\partial\lambda}+(1+\alpha)\mu\Delta\tilde{\phi}=0\\ \tilde{h}-(1+\alpha)\mu\tilde{\psi}=0\end{aligned}\right\}.\tag{19b}$$

The Laplacian operator in the second equation of (19b) has been dropped on the basis of the following argument. Given that the region of integration is the entire sphere, and taking $\chi=\tilde{h}-(1+\alpha)\mu\tilde{\psi}$, we have that $\Delta\chi=0$ implies $\chi=0$; that is, the nonsingular homogeneous solution to $\Delta\psi=0$ is a constant and since the perturbation height vanishes at the pole, this constant must vanish in order that $\tilde{v}\cong\sigma^{-1}\tilde{\psi}_\lambda$ will be nonsingular at the pole. In the remainder of this paper, we refer to (19b) as the *approximate* model.

3. EXACT NORMAL MODE WAVE PROPAGATION THEORY FOR AN ATMOSPHERE IN CONSTANT ROTATION

We have obtained an *exact* perturbation system for atmospheric motions in the presence of arbitrary zonal winds (i.e. (9)–(12)) and by scale analysis an approximate system (i.e. (11) and (19b)). We shall discuss in this section the solutions of the exact equation for a special case, that is when U/σ, the angular zonal wind, is constant. For this special case, the system (9)–(12), describing exactly linearized hydrostatic atmospheric motions, becomes tractable without further approximation. Solutions to the exact equations are formally obtained in this section as a sum over the discrete Hough function normal modes tabulated by LH. The coefficients of the expansion are obtained by solving a Sturm-Liouville equation in z. In the next section, the approximate solution obtained from (19b) is compared with the results of the exact solution so as to gain a better understanding than can be given by scale analysis alone of the errors which may occur in using equation (19b).

Assume now that the angular rotation of the atmosphere and hence α, defined by (18), is constant, that eddies are independent of time, and that forcing occurs only at boundaries. Let subscripts denote partial differentiation with respect to the subscript. New nondimensional dependent variables are now defined using $(2\Omega)^{-1}(1+\alpha)^{-1}$ rather than $(2\Omega)^{-1}$ as characteristic time and are written without tildes. That is, let

$$\tilde{u}=(1+\alpha)u,\ \tilde{v}=(1+\alpha)v,\ \tilde{\psi}=(1+\alpha)\psi,\ h=(1+\alpha)^2h,$$
$$\tilde{w}=(1+\alpha)e^{z/2}Y,\ \tilde{S}=(1+\alpha)^2S.$$

The strength of the mean wind will be described by the nondimensional parameter $\epsilon=\alpha/2(1+\alpha)$. We then may write (9)–(12) as

$$\epsilon u_\lambda-\mu v+\sigma^{-1}h_\lambda=0 \tag{20}$$

$$\epsilon v_\lambda+\mu u+h_\varphi=0 \tag{21}$$

$$\epsilon h_{z\lambda}+Se^{z/2}Y=0 \tag{22}$$

$$\sigma^{-1}u_\lambda+\sigma^{-1}(\sigma v)_\varphi+e^{z/2}(Y_z-\tfrac{1}{2}Y)=0. \tag{23}$$

The parameter S is taken to depend only on z. We shall assume at some isobaric surface, $z=z^0$ (which may be taken to be a bottom boundary), that a boundary condition of the form $Y=W_0\ (\lambda,\varphi)$ is imposed, and that at an open upper boundary a radiation condition is imposed. We now separate variables in the system (20)–(23) by expanding solutions into sums over latitudinal eigenfunctions. To do this, consider the system of differential equations

$$\left.\begin{aligned}\epsilon u_\lambda-\mu v+\sigma^{-1}h_\lambda=0\\ \epsilon v_\lambda+\mu u+h_\varphi=0\\ \sigma^{-1}u_\lambda+\sigma^{-1}(\sigma v)_\varphi+\gamma\epsilon h_\lambda=0\end{aligned}\right\}. \tag{24}$$

For fixed ϵ and for solutions regular at $\mu=\pm1$, (24) defines an eigenvalue problem which has solutions for $h=h(\lambda,\varphi)$, only for a discrete infinity of values of the eigenparameter γ. Assume that h is given by the product of an eigenfunction of (24) and a function of only z. Substitution of (24) into (20)–(23) then separates variables and gives the z dependence of solutions to be defined by the Sturm-Liouville system

$$\left.\begin{aligned}\epsilon h_{z\lambda}+Se^{z/2}Y=0\\ Y_z-\tfrac{1}{2}Y-\gamma\epsilon e^{-z/2}h_\lambda=0\end{aligned}\right\}. \tag{25}$$

We expand the boundary condition $W_0(\lambda,\varphi)$ and the solution $Y(\lambda,\varphi,z)$ as a sum over normal mode solutions to (24)

$$\begin{matrix}W_0\\ Y\end{matrix}=\sum_{m=-\infty}^{\infty}e^{im\lambda}\sum_j\begin{matrix}W_j^m\\ Y_j^m(z,\gamma_j^m)\end{matrix}H_j^m(\mu,\gamma_j^m) \tag{26}$$

where $H_j^me^{im\lambda}$ is an eigenfunction of (24) taken to be an equation for h, with eigenvalue denoted γ_j^m, and $\sum_j$ indicates summation over all eigenfunctions for a given m. Then by (25) the coefficient $Y_j^m(z)$ satisfies

$$Y_{j,zz}^m+(\gamma_j^mS-\tfrac{1}{4})Y_j^m=0 \tag{27}$$

with boundary condition at $z=z_0$, $Y_j^m(z_0)=W_j^m$.

The normal mode solutions of (24) for h, that is the $H_j^m(\mu,\gamma_j^m)$, are called Hough functions, after Hough [7]. The corresponding eigenvalues may be accurately evaluated and qualitative features of the eigenfunctions may be ascertained with little labor by use of the LH paper. Figure 1 plotted from the LH tables illustrates the $\gamma_j^1(\epsilon)$ wave number one spectrum for $0\leq|\epsilon|<\frac{1}{2}$. We should note here that ϵ multiplied by wave number m corresponds to frequency in the LH theory, with $\epsilon>0$ corresponding to the westward propagating modes of LH. Shown are the first four modes for each branch of the spectrum. As the parameter $|\epsilon|$ ranges from zero to one-

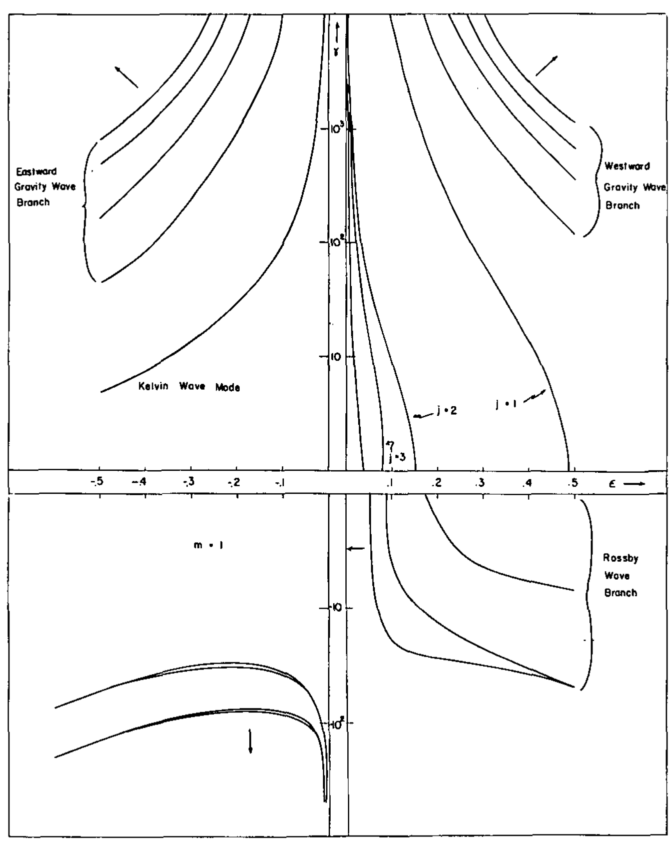

FIGURE 1.—The dependence of the Hough function eigenvalues γ on the zontal wind ϵ for the first four modes of each branch of the spectrum and for $0<\epsilon\leq\frac{1}{2}$. The right hand side shows the spectrum of stationary waves in a westerly flow (traveling westward relative to the mean wind).

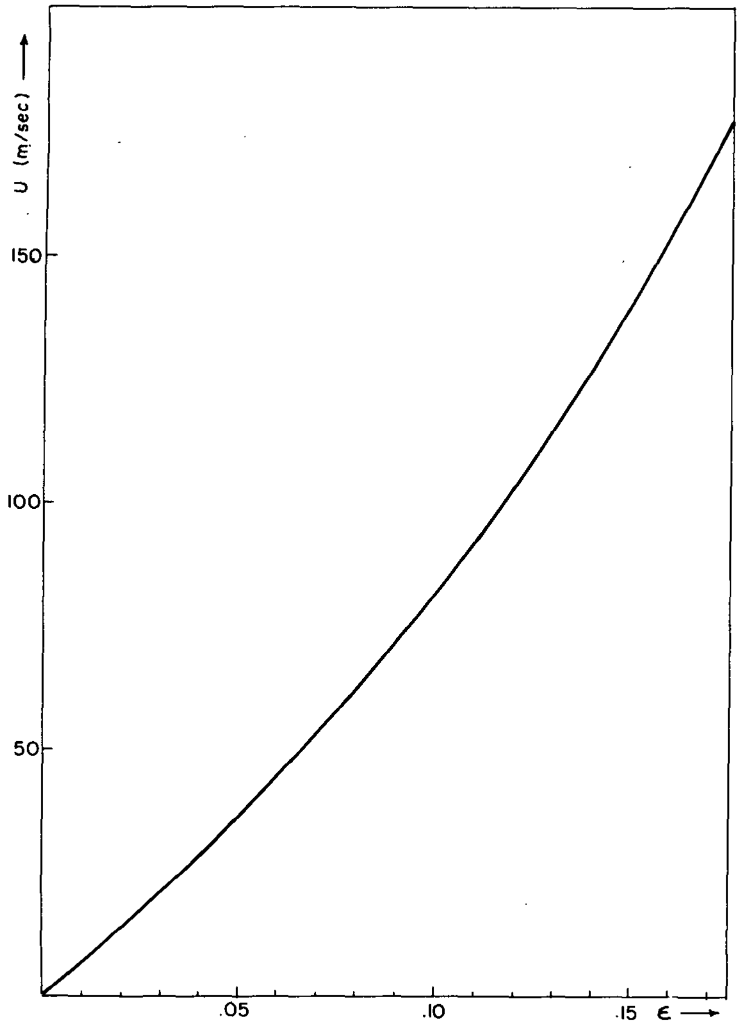

FIGURE 2.—The dependence of dimensional zonal wind at 45°N. on the nondimensional parameter ϵ over the range of values possibly realized in the stratosphere.

half, the solution given by (26) describes waves stationary in longitude propagating vertically through zonal winds ranging in speed from zero to infinity. Figure 2 gives the conversion from ϵ to the mean zonal wind $\bar{u}$ at 45° of lat. It may be shown following LH that for $\epsilon < 1/m$, there are two sets of points in the spectrum of equation (24), a gravity wave and a Rossby wave branch. We shall denote the respective eigensolutions $H^m_{j\,(g)}(\mu)$ and $H^m_{j\,(r)}(\mu)$. The integers $j \geq m$ may be chosen so that as $\epsilon \rightarrow \infty$, $H^m_{j\,(g)}(\mu) \rightarrow P^m_j(\mu)$, and as $\gamma^m \rightarrow 0$, the stream function ψ or σv of the $H^m_{j\,(r)}(\mu)$ mode becomes proportional to $P^m_j(\mu)$, where the $P^m_j(\mu)$ are associated Legendre polynomials. It is revealed by the analysis of LH that as $\gamma^m \rightarrow \infty$, for $\epsilon > 0$ (westerly winds) the $j=m$ Rossby wave mode is not quasi-geostrophic but rather becomes a divergent gravity wave, and for $\epsilon < 0$ (easterly winds) the $j=m$ gravity wave, called the "Kelvin wave," unlike the other gravity waves has geostrophic zonal winds. See LH for further details.

The Hough functions are orthogonal; the boundary conditions may be projected onto the gravity wave and Rossby wave modes by standard procedures. For a given ϵ and for $|m| < 1/|\epsilon|$, the summation in (26) is over both the Rossby wave modes, with eigenvalues denoted $\gamma^m_{j\,(r)}$, $j \geq |m|$, and the gravity wave modes, with eigenvalues denoted $\gamma^m_{j\,(g)}$, $j \geq |m|$, while for $|m| > 1/|\epsilon|$ the sum is over only the gravity waves. (Stationary disturbances with $m = \epsilon^{-1}$ correspond to semidiurnal oscillations in tidal theory.) The amplitude of each mode away from the boundary is then given by the Sturm-Liouville equation (27). Then h is obtained from Y by (25) and u and v from (24) giving

$$\begin{pmatrix} h \\ u \\ v \end{pmatrix} = e^{z}\frac{\partial}{\partial z} e^{-z/2} \sum_{m=-\infty}^{\infty} \sum_{j} (\epsilon \gamma^m_j i m)^{-1} \times \begin{pmatrix} 1 \\ (\mu^2 - m^2\epsilon^2)^{-1}\left(-\mu\sigma\frac{\partial}{\partial \mu} + \epsilon \frac{m^2}{\sigma}\right) \\ (\mu^2 - m^2\epsilon^2)^{-1}\left(-\epsilon\sigma \frac{\partial^2}{\partial\lambda\partial\mu} + \frac{\mu}{\sigma}\frac{\partial}{\partial \lambda}\right) \end{pmatrix} Y^m_j(z) H^m_j(\mu, \gamma^m_j) e^{im\lambda}. \tag{28}$$

This formally completes the exact solution. In the next section we analyze the approximate model and discuss its error.

4. APPROXIMATE NORMAL MODE WAVE PROPAGATION THEORY FOR AN ATMOSPHERE IN CONSTANT ROTATION

In this section we consider the "approximate model," (19b), under the assumptions of section 3 that α is a constant and that motions are independent of time. Then using the definitions preceding (20), (19b) reduces to

$$(\epsilon\Delta + 1)\psi_\lambda - \mu e^{z/2}(Y_z - \tfrac{1}{2}Y) = 0 \tag{29}$$

$$\mu\psi - h = 0. \tag{30}$$

Equations (29) and (30) are to be considered an approximation to (20), (21), and (23). The system (29), (30) is closed with (22). We now again express solutions as sums over latitudinal eigenfunctions.

Let $K^m_j(\mu, \hat{\gamma}^m_j(\epsilon))$ be an eigenfunction of the equation

$$[(1-\mu^2)\psi_\mu]_\mu - \frac{m^2}{1-\mu^2}\psi + \frac{1}{\epsilon}\psi - \mu^2\gamma\psi = 0 \tag{31}$$

with eigenvalues denoted $\gamma = \hat{\gamma}^m_j(\epsilon)$. We separate variables for the systems (22), (29), and (30) by expanding ψ and Y in the eigenfunctions of (31).

$$\begin{pmatrix} \psi \\ Y \end{pmatrix} = \sum_{m=-\infty}^{\infty} e^{im\lambda} \sum_{j} K^m_j(\mu, \hat{\gamma}^m_j) \begin{pmatrix} \Psi^m_j(z) \\ \mu \hat{Y}^m_j(z) \end{pmatrix}. \tag{32}$$

Substituting (32) into (29), and eliminating ψ and h by (30) and (22) reduces (29) to

$$\hat{Y}^m_{j,zz}+(\hat{\gamma}^m_j S-\tfrac{1}{4})\hat{Y}^m_j=0 \tag{33}$$

which, except for the constant $\hat{\gamma}^m_j$, is the same as equation (27). We then obtain from (29) and (31) that

$$\mu\Psi^m_j=(\epsilon\gamma^m_j im)^{-1}e^{z/2}\left(\frac{\partial}{\partial z}-\frac{1}{2}\right)\hat{Y}^m_j(z). \tag{34}$$

Using (13) and (14) we obtain ψ the nondivergent component of the perturbation wind, which is total wind with error $O(\delta^{1/2})$.

The $K^m_j(\mu, \hat{\gamma}^m_j)$ are prolate spheroidal wave functions for $\hat{\gamma}^m_j>0$, and oblate spheroidal wave functions for $\hat{\gamma}^m_j<0$ (cf. Abramowitz et al. [1], p. 752). We are here interested in propagating modes, for which necessarily $\hat{\gamma}^m_j>0$, so we need consider only the prolate spheroidal wave solutions. We identify each $K^m_j(\mu, \hat{\gamma}^m_j)$ with a corresponding Hough function $H^m_j(\mu, \gamma^m_j)$ by the fact that K^m_j and the stream function of the H^m_j mode approach the associated Legendre polynomial $P^m_j(\mu)$ as $\hat{\gamma}^m_j\rightarrow 0$, $\gamma^m_j\rightarrow 0$. There is thus a one to one correspondence between the K^m_j andthe Rossby wave mode Hough functions.

Since it is not presently possible to solve the exact system (9)–(12) when the mean wind has horizontal and vertical variability, comparison between the $K^m_j(\mu)$ and $H^m_j(\mu)$, the approximate and exact solutions for winds in constant angular rotation, can be used for guidance in evaluating the possible errors that may result from using the approximate model (19b) for more general wind shears. For the present problem, use of (29) and (30) rather than the exact equations introduces three kinds of errors into the description of vertical propagation of motions, which we denote (a) *distortion*, (b) *transmission*, and (c) *omission errors*.

Distortion errors result from the fact that for identical bottom boundary conditions, the amount of $W_0(\lambda, \varphi)$, the forcing at the lower boundary, that projects onto the H^m_j mode will be somewhat different from that which projects on the K^m_j mode. This we call *amplitude error*. Also, the eigenfunction variables will have somewhat different latitudinal dependence for the approximate model than for the exact model. This we call *shape error*. Transmission errors occur because $\hat{\gamma}^m_j$ differs somewhat from γ^m_j for a fixed zonal wind. Consequently, the cutoff value for westerly winds, above which there will be no energy transmission, will be somewhat different for the approximate versus the exact model. Omission errors occur because some of the gravity wave modes of (24) which do not correspond to any of the modes of (31) may propagate energy vertically.

These errors are now discussed in greater detail:

a) *Distortion errors:* For comparison of the exact and approximate h's, the approximate model relation (19b) is used. It is here written

$$h=\mu\psi. \tag{35}$$

Recalling that the H^m_j, are eigenfunctions for h and the K^m_j eigenfunctions for ψ, we look at the validity of the approximation $\mu K^m_j\cong H^m_j$ in describing the normal mode geopotential height. In figure 3 are sketched the lowest three $m=1$, and lowest two $m=2$, $H^m_j(\mu)$ Hough functions and the corresponding approximating $\mu K^m_j(\mu)$. The H^m_j and the μK^m_j are not normalized to unity, but both functions have been plotted on roughly the same scales. We have assumed in plotting these figures that $\gamma=\hat{\gamma}=10$, which is the lowest value, of γ that allows transmission when $S=0.025$, a typical value of stability for the stratosphere. Hough function values were read from figures 10 and 13 of LH and spheroidal wave functions from Abramowitz et al. [1], p. 766. It is seen that if the two sets of functions had the same normalization, they would be in close agreement for latitudes north of 30°N., but in tropical latitudes when the *exact* height is nonzero, the approximate height is in large error.

Assuming that essentially all boundary forcing occurs in middle and high latitudes, we may infer that the difference in the expansion coefficients between the exact and approximate models will be small. However, even when the *amplitude error* is small, the *shape error* will be large in tropical latitudes. The approximate model will give a satisfactory approximate description of the shape and amplitude of planetary scale normal mode disturbances in middle and high latitudes provided the forcing and disturbance observations are also confined to middle and high latitudes.

b) *Transmission errors:* In table 1 is given for several of the lowest modes the strength of westerly winds at 45°N. which, according to the exact and approximate models, is just sufficient to trap the given mode. Again S is taken to be 0.025, so cutoff occurs for ϵ such that $\gamma>10$. For $\gamma=10$, the exact ϵ is obtained from the LH table 5, and the approximate ϵ from Abramowitz et al. [1], p. 762 *et seq.*

As further indication of the error which may result when the approximate model is used, $\gamma(\epsilon)$ and $\hat{\gamma}(\epsilon)$ for the two lowest wave number one and wave number two normal modes have been obtained from the above men-

TABLE 1.—*The westerly zonal wind in m./sec. at 45°N. beyond which, according to the theory of this paper, the Rossby wave modes are trapped (assuming trapping for $\gamma<10$).*

m	j	The exact model	The approximate model
1	1	1583.0	400.6
1	2	77.2	83.2
1	3	43.2	44.8
1	4	27.0	28.8
2	2	144.9	125.0
2	3	49.2	50.2
2	4	29.6	30.0
3	3	62.4	59.3
3	4	31.9	
4	4	35.5	
5	5	23.1	

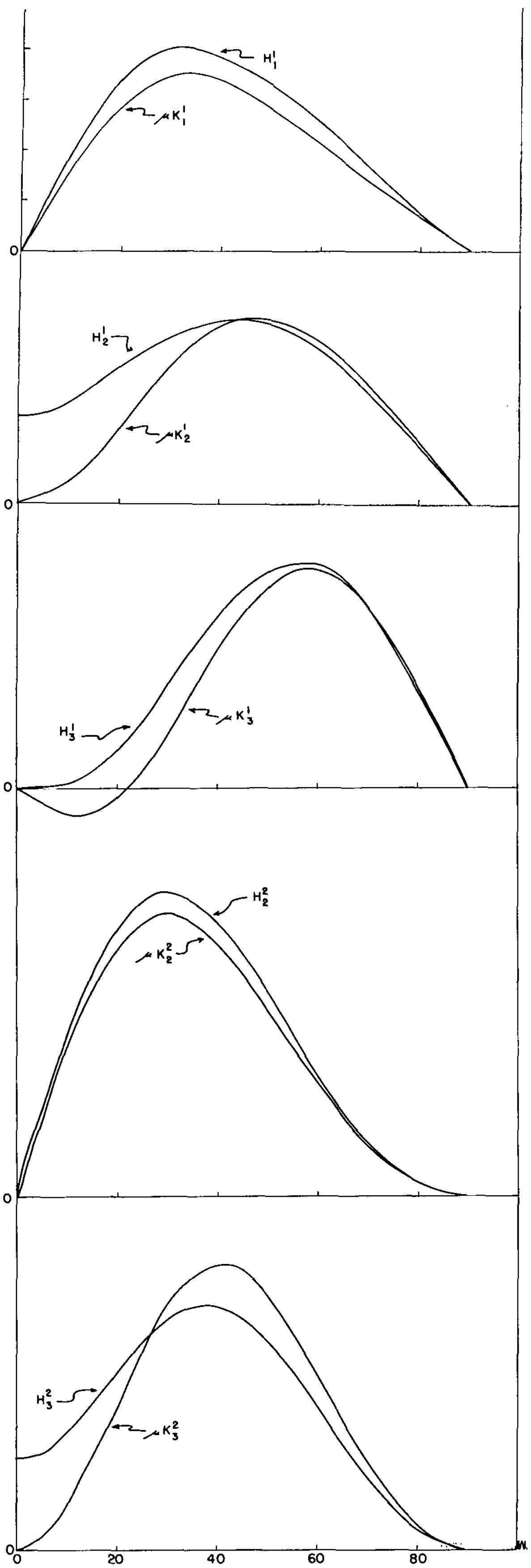

FIGURE 3.—Several of the eigenfunction geopotentials $H_j^m(\mu)$ vs. $\mu K_j^m(\mu)$ $\gamma=\hat{\gamma}=10$. The normalization is arbitrary.

tioned sources and plotted in figure 4. The ratio of an exact to an approximate eigenvalue can be used to measure the error of the approximate eigenvalue. It is seen that the ratio of γ_1^1 to $\hat{\gamma}_1^1$ and the ratio of γ_2^2 to $\hat{\gamma}_2^2$ is close to unity for small ϵ but monotonically increases with ϵ. On the other hand, for $j>m$ the ratio of γ_j^m to $\hat{\gamma}_j^m$ approaches unity for both small and large ϵ; γ_2^1 exceeds $\hat{\gamma}_2^1$ by less than 5 percent for all ϵ. The reasonable agreement between the exact and approximate eigenvalues for $\gamma \lesssim O(\delta^{-1})$, even when the eigenfunctions do not agree in tropical latitudes, can be ascribed to the fact that the term in the horizontal structure equation which may be in error by $O(1)$ within $O(\delta^{1/2})$ of the Equator is then only of amplitude δ compared to other terms in the equation. The deterioration of the approximations for the $j=m$ mode and for $\epsilon >> \delta^{-1}$ is a consequence of the breakdown of the approximate model for this mode as the vertical wavelength becomes much smaller than that assumed by our scaling, and its divergence becomes as large as its vorticity. The higher modes are still satisfactorily described by the approximate model for large ϵ because they remain quasi-nondivergent in nature as their vertical wavelengths become much smaller than is assumed by the scaling. This distinction between the lowest modes and the higher modes manifests itself in the asymptotic behavior of the eigenvalues γ; that is, the $\gamma_j^j(\epsilon)$ decays as $\epsilon^{-1/4}$ like a gravity wave, while the other γ_j^m decay as $\epsilon^{-1/2}$. (See LH.)

c) *Omission errors:* For zonal wind strengths commonly observed in the stratosphere, figure 2 shows that $|\epsilon| \leq 0.15$. It may be inferred from figure 1 that $m=1$ planetary scale stationary gravity waves for westerly winds and for this range of ϵ will have a γ of 10^4, or greater. It follows from the asymptotic formulae of LH that such gravity waves will be trapped within 10° or so from the Equator and will have exponentially small amplitudes in middle and high latitudes. The same conclusion holds for the higher wave number gravity waves. Assuming easterly zonal winds with the strength of the summer stratospheric jet, one finds, however, that the $m=1$ *Kelvin wave* gravity wave will have significantly large amplitude out to middle latitudes.

Since the Rossby wave modes only propagate in westerly winds it follows, according to the constant angular wind theory, that practically all easterly wind stationary wave vertical propagation away from the Equator will be in the Kelvin mode. Gravity wave modes can safely be omitted when studying planetary waves propagating through westerly winds except possibly near the Equator. However, the theory of LH suggests that there may be significant gravity wave mode, stationary wave propagation of planetary scale in easterly zonal winds.

5. APPLICATION TO VERTICAL PROPAGATION OF STRATOSPHERIC PLANETARY WAVES

The normal mode disturbances for an atmosphere in constant angular rotation were discussed in the previous sections. Away from tropical latitudes the eigenfunctions

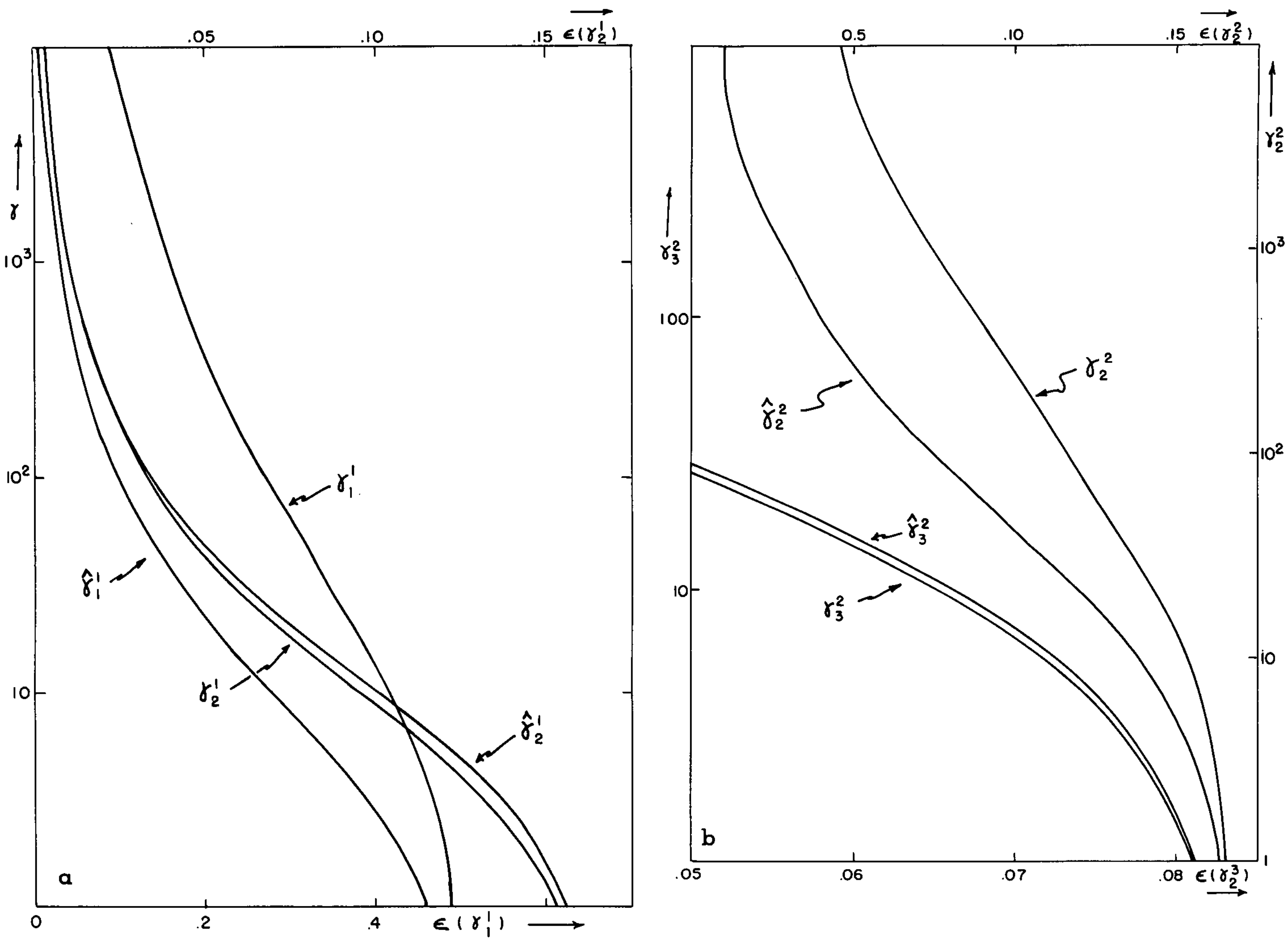

FIGURE 4.—The lowest two wave number one and wave number two exact and approximate eigenvalues vs. the zonal wind ϵ.

of the approximate model (19b) differ little from those obtained from exact theory, except that the approximate model does not contain the gravity wave modes. The approximate model predicts eigenvalues with little error with the exception of those for the lowest latitudinal modes.

The importance of the various possible modes which may propagate into the upper atmosphere is now estimated as follows. Teweles [21] has computed for January 1958 the geopotential height amplitudes at 50 mb. for the first four wave numbers. Given a zonal wind $\bar{u}$ with angular velocity a constant, the wave number geopotential may be expanded in the corresponding Hough functions or the approximating spheroidal wave functions. Hence the modal amplitudes obtained by projection of Teweles' data onto the various vertically propagating Hough function normal modes can be estimated.

For example, assume $\gamma=10$. In figure 5 are shown geopotential height maps obtained by projecting wave number one for January 1958 (taken from Teweles [21]) onto the H_2^1 and H_3^1 Hough function normal modes. One may infer from the eigenfunction figures of LH that the eigenfunctions only change slowly with the mean wind ϵ, so we expect that about the same maps would be obtained for zonal winds varying from 10 to 10^2 m./sec. Referring to table 1, we see that these two modes propagate when $\bar{u}$ at 45°N. is less than 77 and 43 m./sec., respectively. For stronger winds than these cutoff values, these modes will be exponentially attenuated above 50 mb., while for zonal winds weaker than cutoff, the eddy geopotentials and winds of the resulting propagating mode increase above 50 mb. as $p^{-1/2}$. Thus, the present linear theory predicts that at 5 μb. (85 km.) the wave number one eddy geopotential height amplitude should exceed 6 km. for intervening westerly zonal winds of speeds less than 75 m./sec. and should exceed 20 km. when the intervening $\bar{u}>0$ does not exceed 40 m./sec. The corresponding eddy velocities are several hundred m./sec. or greater. Similar amplitudes would be obtained for the $m=2$ modes which propagate. If we assume ϵ and S depend only on z and apply a geometric optics approximation, then comparable eddy wind amplitudes are found provided the local value of the parameter $[\gamma(\epsilon\ (z))S(z)-\frac{1}{4}]$ remains greater than zero everywhere between 50 mb. and 5 μb.

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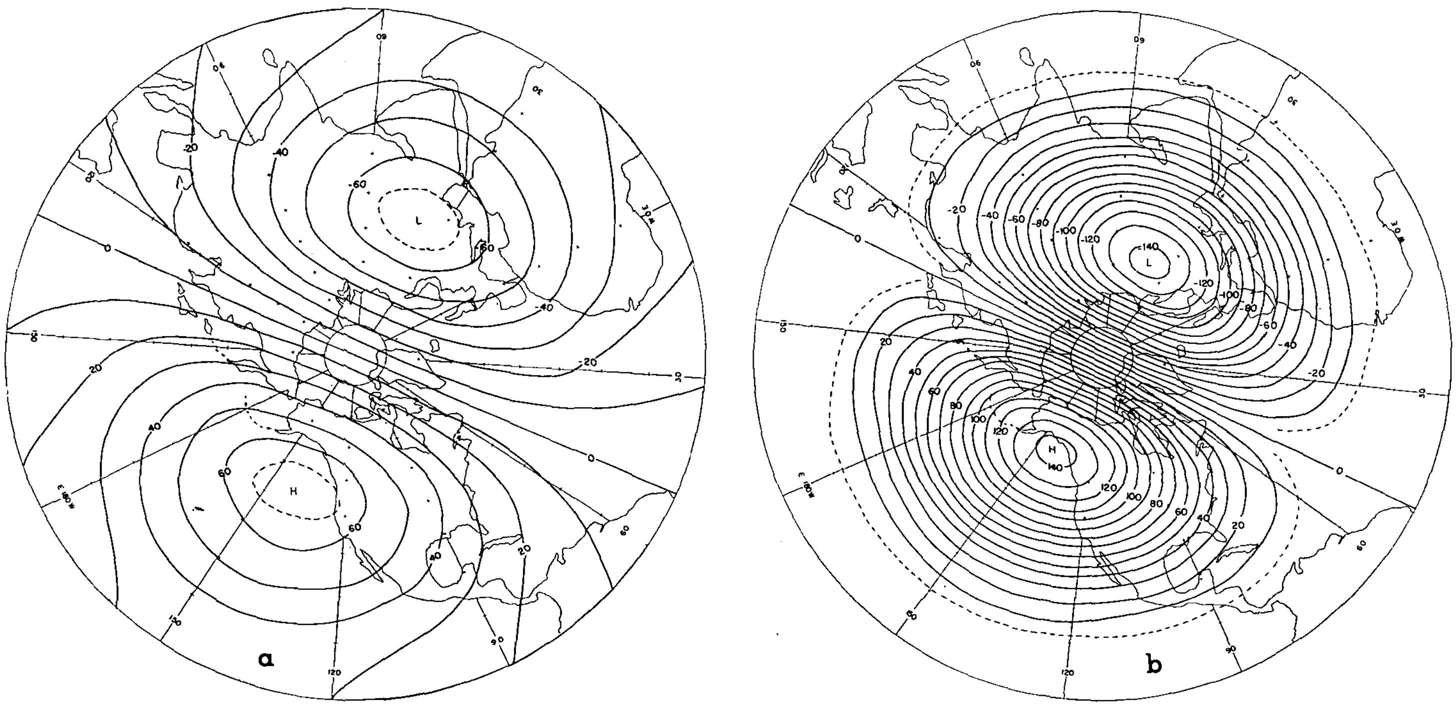

FIGURE 5.—The projection of January 1958, 50-mb. wave number one geopotential onto (a) the $H_2^1(\mu, 10)$ Hough function mode and (b) the $H_3^1(\mu, 10)$ Hough function mode. Units are m.

To continue our results into the Southern Hemisphere, we note that the H_2^1 mode is symmetric and the H_3^1 mode antisymmetric about the Equator. For the computation of amplitudes of these normal modes the Southern Hemisphere January 50-mb. wave number one amplitude has been assumed to be zero, and small phase shifts with latitude of the longitudinal phase have been neglected.

6. CONCLUDING REMARKS

In the preceding section, horizontal wind variations were neglected and the zonal winds were assumed to be westerly at all levels with the observed winter mid-latitude amplitudes of the upper stratosphere. Charney and Drazin [3], in applying their model to disturbances on a spherical earth by expanding disturbances in spherical harmonics, concluded that there would be no propagation for zonal winds greater than 38 m./sec. for modes with j (the degree of the spherical harmonic) three or greater. In comparison, our $j=3$ modes (cf. table 1) have cutoff velocities of 43, 49, and 62 m./sec. Charney and Drazin hypothesized that the escape of large amounts of planetary wave energy could be prevented during the winter by the observed large westerly zonal winds above the tropopause in middle latitudes. However, we find that strong stratospheric winds are not sufficient to trap planetary waves in the lower atmosphere. For an atmosphere in constant angular rotation there exist six planetary wave modes that propagate for $\bar{u}>38$ m./sec. and indeed two modes for $\bar{u}>100$ m./sec. That part of the planetary waves observed in the lower stratosphere which projects onto each of these modes has a height amplitude of some tens of meters and gives corresponding winds of several m./sec. Hence the eddy winds of each propagating mode above some level in the lower thermosphere should be in excess of hundreds of m./sec. Stationary winds with these velocities are, however, not observed at these levels.

The discrepancy between conclusions obtained by the more realistic theory and those obtained by Charney and Drazin may be partly explained by noting that the normal mode planetary waves on a spherical earth "feel" a smaller Coriolis parameter than the middle latitude value assumed by Charney-Drazin and that on a spherical earth, significant disturbances of larger horizontal scale than the scales they assumed may occur. Their Cartesian model indicates that when the Coriolis parameter or horizontal wave number is decreased, stationary planetary waves can propagate through stronger westerly winds. Furthermore, an equatorial β-plane model predicts transmission of planetary waves through westerlies of any strength (Lindzen [9]).

It is clear that if the wave amplitudes approaching those predicted by the present theory for the lower thermosphere were realized, there would be a severe violation of the linearization assumption of our model. However, we believe that the disagreement between observations and the present theory is first of all a result of the neglect of latitudinal wind variations. Latitudinally variable zonal winds can confine a disturbance to limited latitudinal belts, greatly reducing the possible vertical propagation of disturbances (Dickinson [4]). We have also neglected diabatic damping of disturbances, which can be important to lowest order near the stratopause and hence should also necessarily be included for accurate analysis of Rossby waves at this level.

Because of the decrease of the effective latitudinal scale of disturbances by variable zonal winds, it seems likely that most of the low frequency planetary wave energy will be in latitudinal scales comparable to or smaller than the latitudinal scale of zonal wind systems. Provided then such zonal wind systems have a latitudinal scale of $O(\delta^{1/2})$ or less, the latitudinal scale of disturbances can be expected to be $O(\delta^{1/2})$, so that the assumptions made to derive (19) will be vaild. Equations (19b) provide a satisfactory linear model for further study of planetary Rossby waves propagating through actual zonal wind systems away from equatorial latitudes. Winds in equatorial latitudes with a long enough time scale will still be *quasi-geostrophic* but not necessarily *quasi-nondivergent*. The geopotential height h rather than the stream function ψ, then becomes the most convenient variable to use for planetary waves in the presence of wind shears. Using this variable, one can easily generalize the equatorial β-plane model of Lindzen [9] to obtain an equation for equatorial latitudes which describes quasi-geostrophic motions in the presence of mean horizontal and vertical shears.

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